

Seiberg-Witten Theories, Integrable Models and Perturbative Prepotentials

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Abstract

This is a very brief review of relations between Seiberg-Witten theories and integrable systems with emphasis on the perturbative prepotentials presented at the E.S.Fradkin Memorial Conference.

Introductory remarks. Seiberg-Witten theory [1] provides a description of the effective low-energy actions of four dimensional $\mathcal{N} = 2$ SUSY $SU(N)$ Yang-Mills theories in terms of finite-dimensional integrable systems [2]. Such a description has been extended to other $\mathcal{N} = 2$ gauge theories. There are basically three different ways to extend the original Seiberg-Witten theory. First of all, one may consider other gauge groups, from other simple classical groups to those being a product of several simple factors. The other possibility is to add some matter hypermultiplets in different representations. The two main cases here are the matter in fundamental or adjoint representations. At last, the third possible direction to deform Seiberg-Witten theory is to consider 5- or 6-dimensional theories, compactified respectively onto the circle of radius R_5 or torus with modulus R_5/R_6 (if the number of dimensions exceeds 6, the gravity becomes obligatory coupled to the gauge theory). It is important to investigate thoroughly different deformations of Seiberg-Witten theory and corresponding deformations of the proper integrable systems in order to better establish the correspondence between these two. In fact, since repeating the whole original Seiberg-Witten procedure, which might unquestionably prove the correspondence, is technically quite tedious for theories with more vacuum moduli, in most cases the procedure of identification of the gauge theory and integrable model comes via comparing the three characteristics:

- deformations of the two
- number of the vacuum moduli and external parameters
- perturbative prepotentials

The number of vacuum moduli (i.e. the number of scalar fields that may have non-zero v.e.v.'s) on the physical side should be compared with dimension of the moduli space of the spectral curves in integrable systems, while the external parameters in gauge theories (hypermultiplet masses) should be also some external parameters in integrable models (coupling constants, values of Casimir functions for spin chains etc). As for third item, the low energy effective action in Seiberg-Witten theory is described in terms of *prepotentials* that celebrate a lot of properties familiar from the original studies of pure topological theories (where have been neglected the possibility of the light excitations to move). These

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properties include the identification [2, 3, 4] of prepotentials as quasiclassical (Whitham) τ -functions [5, 6] and peculiar equations, of which the (generalized) WDVV equations [7, 6, 8] are the best known example. In the prepotential, the contributions of particles and solitons/monopoles (dyons) sharing the same mass scale, are still distinguishable, because of different dependencies on the bare coupling constant, *i.e.* on the modulus τ of the bare coordinate elliptic curve (in the UV-finite case) or on the Λ_{QCD} parameter (emerging after dimensional transmutation in UV-infinite cases). In the limit $\tau \rightarrow i\infty$ ($\Lambda_{QCD} \rightarrow 0$), the solitons/monopoles do not contribute and the prepotential reduces to the “perturbative” one, describing the spectrum of non-interacting *particles*. It is immediately given by the SUSY Coleman-Weinberg formula [9]:

$$\mathcal{F}_{pert}(a) = \sum_{\text{reps } R, i} (-)^F \text{Tr}_R(a + M_i)^2 \log(a + M_i) \quad (1)$$

Seiberg-Witten theory (actually, the identification of appropriate integrable system) can be used to construct the non-perturbative prepotential, describing the mass spectrum of all the “light” (non-stringy) excitations (including solitons/monopoles). Switching on Whitham times [4] presumably allows one to extract some correlation functions in the “light” sector. Now note that the problem of calculation of the prepotential in physical theory is simple only at the perturbative level, where it is given just by the leading contribution, since the β -function in $\mathcal{N} = 2$ theories is non-trivial only in one loop. However, the calculation of higher (instantonic) corrections in the gauge theory can be hardly done at the moment¹. Therefore, the standard way of doing is to rely on integrable calculations. This is why the establishing the “gauge theories \leftrightarrow integrable theories” correspondence is of clear practical (apart from theoretical) importance.

Integrability of Seiberg-Witten theory . Let us describe now the correspondence in more details. The most important result of [1], from this point of view, is that the moduli space of vacua and low energy effective action in SYM theories are completely given by the following input data:

- Riemann surface \mathcal{C}
- moduli space \mathcal{M} (of the curves \mathcal{C})
- meromorphic 1-form dS on \mathcal{C}

How it was pointed out in [2, 9], this input can be naturally described in the framework of some underlying integrable system. To this end, first, we introduce bare spectral curve E that is torus $y^2 = x^3 + g_2x^2 + g_3$ for the UV-finite² gauge theories with the associated holomorphic 1-form $d\omega = dx/y$. This bare spectral curve degenerates into the double-punctured sphere (annulus) for the asymptotically free theories (where dimensional transmutation occurs): $x \rightarrow w + 1/w$, $y \rightarrow w - 1/w$, $d\omega = dw/w$. On this bare curve, there are given either a matrix-valued Lax operator $L(x, y)$ if one considers an extension of the $N \times N$ Toda Lax representation, or another matrix Lax operator $\mathcal{L}_i(x, y)$ associated with an extension of the 2×2 Toda Lax representation and defining the transfer matrix $T(x, y)$. The corresponding dressed spectral curve \mathcal{C} is defined either from the formula

¹In order to check the very ideology that the integrable systems lead to the correct answers, there were calculated first several corrections [10]. The results proved to exactly coincide with the predictions obtained within the integrable approach.

²The situation is still unclear in application to the case of fundamental matter with the number of matter hypermultiplets equal to $N_f = 2N$. In existing formulation for spin chains the bare coupling constant appears rather as a twist in gluing the ends of the chain together [11] (this parameter occurs only when $N_f = 2N$) and is not immediately identified as a modulus of a *bare* elliptic curve. This problem is a fragment of a more general puzzle: spin chains have not been described as Hitchin systems; only the “ 2×2 ” Lax representation is known, while its “dual” $N_c \times N_c$ one is not yet available.

$\det(L - \lambda) = 0$, or from $\det(T - w) = 0$. This spectral curve is a ramified covering of E given by the equation

$$\mathcal{P}(\lambda; x, y) = 0 \quad (2)$$

In the case of the gauge group $G = SU(n)$, the function \mathcal{P} is a polynomial of degree n in λ . Thus, we have the spectral curve \mathcal{C} the moduli space \mathcal{M} of the spectral curve given just by coefficients of \mathcal{P} . The third important ingredient of the construction is the generating 1-form $dS \cong \lambda d\omega$ meromorphic on \mathcal{C} (“ \cong ” denotes the equality modulo total derivatives). From the point of view of the integrable system, it is just the shortened action “ pdq ” along the non-contractible contours on the Hamiltonian tori. Its defining property is that the derivatives of dS with respect to the moduli (ramification points) are holomorphic differentials on the spectral curve. The prepotential \mathcal{F} and other “physical” quantities are defined in terms of the cohomology class of dS :

$$\begin{aligned} a_i &= \oint_{A_i} dS, \\ a_i^D &\equiv \frac{\partial \mathcal{F}}{\partial a_i} = \oint_{B_i} dS, \\ A_I \circ B_J &= \delta_{IJ}. \end{aligned} \quad (3)$$

The first identity defines here the appropriate flat moduli, while the second one – the prepotential. The defining property of the generating differential dS is that its derivatives w.r.t. moduli give holomorphic 1-differentials. In particular,

$$\frac{\partial dS}{\partial a_i} = d\omega_i \quad (4)$$

and, therefore, the second derivative of the prepotential w.r.t. a_i ’s is the period matrix of the curve \mathcal{C} (physically charges, i.e. effective coupling constants in the gauge theory):

$$\frac{\partial^2 \mathcal{F}}{\partial a_i \partial a_j} = T_{ij} \quad (5)$$

The latter formula allows one to identify prepotential with logarithm of the τ -function of the Whitham hierarchy [5, 6, 4]: $\mathcal{F} = \log \tau$. So far we reckoned without matter hypermultiplets. In order to include them, one just needs to consider the surface \mathcal{C} with punctures. Then, the masses are proportional to the residues of dS at the punctures, and the moduli space has to be extended to include these mass moduli. All other formulas remain in essence the same (see [9] for more details).

Seiberg-Witten theories vs. integrable systems. The table of known relations between gauge theories and integrable systems is drawn in Fig.2. The original

Seiberg-Witten model, which is the $4d$ pure gauge $SU(N)$ theory (in fact, in their papers [1], the authors considered the $SU(2)$ case only, but the generalization made in [12] is quite immediate), is the upper left square of the table. The remaining part of the table contains possible deformations. Here only two of the three possible ways to deform the original Seiberg-Witten model are shown. Otherwise, the table would be three-dimensional. In fact, the third direction related to changing the gauge group, although being of an interest is slightly out of the main line. Therefore, we only make several comments on it. In particular, right here note that the generalization of the original Seiberg-Witten theory to another simple group G is quite immediate and corresponds to the Toda chain for the dual affine algebra \hat{G}^\vee [13]. One direction in the table corresponds to matter hypermultiplets added. The most interesting is to add

SUSY physical theory	Pure gauge SYM theory, gauge group G	SYM theory with fund. matter	SYM theory with adj. matter
4d	inhomogeneous periodic Toda chain for the dual affine \hat{G}^\vee (<i>non-periodic Toda chain</i>)	elliptic Calogero model (<i>trigonometric Calogero model</i>)	inhomogeneous periodic XXX spin chain (<i>non-periodic chain</i>)
5d	periodic relativistic Toda chain (<i>non-periodic chain</i>)	elliptic Ruijsenaars model (<i>trigonometric Ruijsenaars</i>)	periodic XXZ spin chain (<i>non-periodic chain</i>)
6d	periodic “Elliptic” Toda chain (<i>non-periodic chain</i>)	Dell system (<i>dual to elliptic Ruijsenaars, elliptic-trig.</i>)	periodic XYZ (elliptic) spin chain (<i>non-periodic chain</i>)

Figure 1: SUSY gauge theories \iff integrable systems correspondence. The perturbative limit is marked by the italic font (in parenthesis).

matter in adjoint or fundamental representations, although attempts to add antisymmetric and symmetric matter hypermultiplets were also done (see [14] for the construction of the curves and [15] for the corresponding integrable systems). Adding matter in other representations in the basic $SU(N)$ case leads to non-asymptotically free theories. Columns: Matter in adjoint *vs.* fundamental representations of the gauge group. Matter in adjoint representation can be described in terms of a larger pure SYM model, either with higher SUSY or in higher space-time dimension. Thus models with adjoint matter form a hierarchy, naturally associated with the hierarchy of integrable models $Toda\ chain \hookrightarrow Calogero \hookrightarrow Ruijsenaars \hookrightarrow Dell$ [2, 16, 17, 3, 18, 19, 20, 21]³. Similarly, the models with fundamental matter are related to the hierarchy of spin chains originated from the Toda chain: $Toda\ chain \hookrightarrow XXX \hookrightarrow XXZ \hookrightarrow XYZ$ [23, 24, 25, 26]⁴. Note that, while coordinates in integrable systems describing pure gauge theories and those with fundamental matter, live on the cylinder (i.e. the dependence on coordinates is trigonometric), the coordinates in the Calogero system (adjoint matter added) live on a torus⁵. However, when one takes the perturbative limit, the coordinate dependence becomes trigonometric. Lines: Gauge theories in different dimensions. Integrable systems relevant for the description of vacua of $d = 4$ and $d = 5$ models are respectively the Calogero and Ruijsenaars ones (which possess the ordinary Toda chain and “relativistic Toda chain” as Inosemtsev’s limits [29]), while $d = 6$ theories are described by the double elliptic (Dell) systems. When we go from $4d$ (Toda, XXX, Calogero) theories to $5d$ (relativistic Toda, XXZ, Ruijsenaars) theories the momentum-dependence of the Hamiltonians becomes trigonometric instead

³The generalization of the models of this class to other groups can be found in [22].

⁴The other classical groups for spin chains/fundamental matter theories were considered in [27]. Another possibility, the gauge groups which are products of simple factors and bi-fundamental matter, were proposed in [28], while the corresponding integrable models discussed in [11].

⁵Since these theories are UV finite, they depend on an additional (UV-regularizing) parameter, which is exactly the modulus τ of the torus.

or rational. Similarly, the $6d$ theories give rise to an elliptic momentum-dependence of the Hamiltonians. Since adding the adjoint hypermultiplet elliptizes the coordinate dependence, the integrable system corresponding to $6d$ theory with adjoint matter celebrates both the coordinate- and momentum-dependencies elliptic. A candidate for “the elliptic Toda chain” was proposed in [30].

Prepotentials in Seiberg-Witten and integrable theories. We already discussed the important role of the prepotentials, in particular, of their perturbative part. This latter helps one to make an identification of the integrable theory with the gauge theory. Note that many crucial properties of the prepotential are seen already at the perturbative level (say, the WDVV equations [9, 31]). Therefore, now we are going to give an explicit description of the perturbative prepotentials for the Seiberg-Witten theories listed in the previous paragraph. The technical tool that allows one to proceed with effective perturbative expansion of the prepotential is “the residue formula”, the variation of the period matrix, that is the third derivatives of $\mathcal{F}(a)$ (see, e.g. [9]):

$$\frac{\partial^3 \mathcal{F}}{\partial a_i \partial a_j \partial a_k} = \frac{\partial T_{ij}}{\partial a_k} = \frac{1}{2\pi i} \text{res}_{d\xi=0} \frac{d\omega_i d\omega_j d\omega_k}{\delta dS}, \quad (6)$$

where $\delta dS \equiv d\left(\frac{dS}{d\xi}\right) d\xi$. We remark that although $d\xi$ does not have zeroes on the *bare* spectral curve when it is a torus or doubly punctured sphere, it does in general however possess them on the covering \mathcal{C} . In order to construct the perturbative prepotentials, one merely can note that, at the leading order, the Riemann surface (spectral curve of the integrable system) becomes rational. Therefore, the residue formula allows one to obtain immediately the third derivatives of the prepotential as simple residues on the sphere. In this way, one can check that the prepotential has actually the form (1) (see details in [9]). As a concrete example, let us consider the $SU(n)$ gauge group. Then, say, perturbative prepotential for the pure gauge theory acquires the form

$$\mathcal{F}_{\text{pert},V} = \frac{1}{4} \sum_{ij} (a_i - a_j)^2 \log(a_i - a_j) \quad (7)$$

This formula establishes that when v.e.v.’s of the scalar fields in the gauge supermultiplet are non-vanishing (perturbatively a_r are eigenvalues of the vacuum expectation matrix $\langle\phi\rangle$), the fields in the gauge multiplet acquire masses $m_{rr'} = a_r - a_{r'}$ (the pair of indices (r, r') label a field in the adjoint representation of G). In the $SU(n)$ case, the eigenvalues are subject to the condition $\sum_i a_i = 0$. Analogous formula for the adjoint matter contribution to the prepotential is

$$\mathcal{F}_{\text{pert},A} = -\frac{1}{4} \sum_{ij} (a_i - a_j + M)^2 \log(a_i - a_j + M) \quad (8)$$

while the contribution of one fundamental matter hypermultiplet reads as

$$\mathcal{F}_{\text{pert},F} = -\frac{1}{4} \sum_i (a_i + m)^2 \log(a_i + m) \quad (9)$$

Similar formulas can be obtained for the other groups.⁶ The perturbative prepotentials in $4d$ theories with fundamental matter are discussed in detail in [9] (see also [32]). The

⁶The eigenvalues of $\langle\phi\rangle$ in the first fundamental representation of the classical series of the Lie groups are

$$\begin{aligned} B_n (SO(2n+1)) : & \quad \{a_1, \dots, a_n, 0, -a_1, \dots, -a_n\}; \\ C_n (Sp(n)) : & \quad \{a_1, \dots, a_n, -a_1, \dots, -a_n\}; \\ D_n (SO(2n)) : & \quad \{a_1, \dots, a_n, -a_1, \dots, -a_n\} \end{aligned} \quad (10)$$

formulas presented above in the $4d$ case can be almost immediately extended to the “relativistic”⁷ $5d$ cal N2 SUSY gauge models with one compactified dimension. One can understand the reason for this “relativization” in the following way. Considering four plus one compact dimensional theory one should take into account the contribution of all Kaluza-Klein modes to each 4-dimensional field. Roughly speaking it leads to the 1-loop contributions to the effective charge of the form⁸

$$T_{ij} \sim \sum_{\text{masses}} \log \text{ masses} \sim \sum_m \log \left(a_{ij} + \frac{m}{R_5} \right) \sim \log \prod_m (R_5 a_{ij} + m) \sim \log \sinh R_5 a_{ij} \quad (13)$$

i.e. coming from $4d$ to $5d$ one should make a substitution $a_{ij} \rightarrow \sinh R_5 a_{ij}$, at least, in the formula for the perturbative prepotential. Now the same general argument applied to the $5d$ case, can be equally applied to the $6d$ case, or to the theory with *two* extra compactified dimensions, of radii R_5 and R_6 . Indeed, the account of the Kaluza-Klein modes allows one to predict the perturbative form of charges in the $6d$ case as well. Namely, one should expect them to have the form⁹

$$\begin{aligned} T_{ij} &\sim \sum_{\text{masses}} \log \text{ masses} \sim \sum_{m,n} \log \left(a_{ij} + \frac{m}{R_5} + \frac{n}{R_6} \right) \sim \\ &\sim \log \prod_{m,n} \left(R_5 a_{ij} + m + n \frac{R_5}{R_6} \right) \sim \log \theta_* \left(R_5 a_{ij} \left| i \frac{R_5}{R_6} \right. \right) \end{aligned} \quad (14)$$

i.e. coming from $4d$ ($5d$) to $6d$ one should replace the rational (trigonometric) expressions by the elliptic ones, at least, in the formulas for the perturbative prepotential, the (imaginary part of) modular parameter being identified with the ratio of the compactification radii R_5/R_6 . More rigid derivations using the residue formula confirms these results. In the fundamental matter case, they look as follows [9, 25, 26]. The spectral curve in all cases can be written in the form

$$w + \frac{Q^{(d)}(\xi)}{w} = 2P^{(d)}(\xi) \quad (15)$$

or

$$W + \frac{1}{W} = \frac{2P^{(d)}(\xi)}{\sqrt{Q^{(d)}(\xi)}}, \quad W \equiv \frac{w}{\sqrt{Q^{(d)}(\xi)}} \quad (16)$$

while the eigenvalues in the adjoint representation have the form

$$\begin{aligned} B_n : \quad & \{\pm a_j; \pm a_j \pm a_k\}; \quad j < k \leq n \\ C_n : \quad & \{\pm 2a_j; \pm a_j \pm a_k\}; \quad j < k \leq n \\ D_n : \quad & \{\pm a_j \pm a_k\}, \quad j < k \leq n \end{aligned} \quad (11)$$

Analogous formulas can be written for the exceptional groups too. The prepotential in the pure gauge theory can be read off from the formula (11) and has the form

$$\begin{aligned} B_n : \quad \mathcal{F}_{\text{pert}} &= \frac{1}{4} \sum_{i,i} \left((a_i - a_j)^2 \log(a_i - a_j) + (a_i + a_j)^2 \log(a_i + a_j) \right) + \frac{1}{2} \sum_i a_i^2 \log a_i; \\ C_n : \quad \mathcal{F}_{\text{pert}} &= \frac{1}{4} \sum_{i,i} \left((a_i - a_j)^2 \log(a_i - a_j) + (a_i + a_j)^2 \log(a_i + a_j) \right) + 2 \sum_i a_i^2 \log a_i; \\ D_n : \quad \mathcal{F}_{\text{pert}} &= \frac{1}{4} \sum_{i,i} \left((a_i - a_j)^2 \log(a_i - a_j) + (a_i + a_j)^2 \log(a_i + a_j) \right) \end{aligned} \quad (12)$$

⁷i.e. with trigonometric momentum dependence

⁸Hereafter, we denote $a_{ij} \equiv a_i - a_j$.

⁹Hereafter, θ_* denotes the θ -function with odd characteristics.

in the perturbative limit, only the first term in the r.h.s. of these formulas survives. The generating differential dS is always of the form

$$dS = \xi d \log W \quad (17)$$

The concrete forms of the functions introduced here are:

$$Q^{(4)}(\xi) \sim \prod_{\alpha}^{N_f} (\xi - m_{\alpha}), \quad Q^{(5)}(\xi) \sim \prod_{\alpha}^{N_f} \sinh(\xi - m_{\alpha}), \quad Q^{(6)}(\xi) \sim \prod_{\alpha}^{N_f} \frac{\theta_*(\xi - m_{\alpha})}{\theta_*^2(\xi - \xi_i)} \quad (18)$$

$$P^{(4)} \sim \prod_i^N (\xi - a_i), \quad P^{(5)} \sim \prod_i^N \sinh(\xi - a_i), \quad P^{(6)} \sim \prod_i^N \frac{\theta_*(\xi - a_i)}{\theta_*(\xi - \xi_i)} \quad (19)$$

(in $P^{(5)}(\xi)$, there is also some exponential of ξ unless $N_f = 2N$, see [26]), ξ_i 's being just external parameters. The perturbative part of the prepotential can be calculated using these manifest expressions and the residue formula and is always of the form¹⁰

$$\mathcal{F}_{pert} = \frac{1}{4} \sum_{i,j} f^{(d)}(a_{ij}) - \frac{1}{4} \sum_{i,\alpha} f^{(d)}(a_i - m_{\alpha}) + \frac{1}{16} \sum_{\alpha,\beta} f^{(d)}(m_{\alpha} - m_{\beta}) \quad (20)$$

The explicit form of these functions is

$$\begin{aligned} f^{(4)}(x) &= x^2 \log x, \quad f^{(5)}(x) = \sum_n f^{(4)}\left(x + \frac{n}{R_5}\right) = \frac{1}{3} |x^3| - \frac{1}{2} \text{Li}_3\left(e^{-2|x|}\right), \\ f^{(6)}(x) &= \sum_{m,n} f^{(4)}\left(x + \frac{n}{R_5} + \frac{m}{R_6}\right) = \sum_n f^{(5)}\left(x + n \frac{R_5}{R_6}\right) = \\ &= \left(\frac{1}{3} |x^3| - \frac{1}{2} \text{Li}_{3,q}\left(e^{-2|x|}\right) + \text{quadratic terms}\right) \end{aligned} \quad (21)$$

so that

$$f^{(4)''} = \log x, \quad f^{(5)''}(x) = \log \sinh x, \quad f^{(6)''}(x) = \log \theta_*(x) \quad (22)$$

Note that, in the $6d$ case, N_f is always equal to $2N$,¹¹ and, in $d = 5, 6$, there is a restriction $\sum a_i = \sum \xi_i = \frac{1}{2} \sum m_{\alpha}$ which implies that the gauge moduli would be rather associated with a_i shifted by the constant $\frac{1}{2N} \sum m_{\alpha}$. In these formulas, $\text{Li}_3(x)$ is the tri-logarithm, while $\text{Li}_{3,q}(x)$ is the elliptic tri-logarithm [33]. Now let us consider the adjoint matter case¹². In this case, using the previously developed general arguments, one would expect for the perturbative prepotential to be of the form

$$\mathcal{F}_{pert} = \frac{1}{4} \sum_{i,j} f^{(d)}(a_{ij}) - \frac{1}{4} \sum_{i,j} f^{(d)}(a_{ij} + M) \quad (24)$$

¹⁰The term in the prepotential that depends only on masses is not essential for the standard Seiberg-Witten theory but is crucial for the prepotential to enjoy its main properties, similar to the WDVV equations. This term is unambiguously restored from the residue formula.

¹¹Otherwise, the function in the r.h.s. of formula (16) is not elliptic.

¹²The prepotential in the pure gauge theory is described by the first term in (20) in $d = 4$, while in $d = 5$ it gets an additional cubic term

$$\begin{aligned} \mathcal{F}_{pert} &= \frac{1}{4} \sum_{i,j} \left(\frac{1}{3} a_{ij}^3 - \frac{1}{2} \text{Li}_3\left(e^{-2a_{ij}}\right) \right) + \frac{N_c}{2} \sum_{i>j>k} a_i a_j a_k = \\ &= \frac{1}{4} \sum_{i,j} \left(\frac{1}{3} a_{ij}^3 - \frac{1}{2} \text{Li}_3\left(e^{-2a_{ij}}\right) \right) + \frac{N_c}{6} \sum_i a_i^3 \end{aligned} \quad (23)$$

Because of the requirement $N_f = 2N$, it is still unclear how to deal with the pure gauge ($N_f = 0$) theory in $d = 6$.

however, the calculations in this case are more complicated. In particular, the spectral curves are far more involved (see [17, 34, 18, 19, 21]). Nevertheless, in the perturbative limit they simplify drastically and coincide with the perturbative curves for the fundamental matter case with the hypermultiplet masses pairwise identified and equal to $a_i + M$, with the generating 1-form dS being of the same form (17). This immediately leads to the result (24) [34, 35, 18, 20, 21]. I am grateful to H.W.Braden, A.Gorsky, A.Marshakov and A.Morozov for numerous discussions and T.Takebe for kind hospitality at the Ochanomizu University, Tokyo, where this work was completed. The work is partly supported by the grants: RFBR-00-02-16477-a, INTAS 99-0590, CRDF grant #6531 and the JSPS fellowship for research in Japan.

References

- [1] N.Seiberg and E.Witten, Nucl.Phys., **B426** (1994) 19, hep-th/9407087; Nucl.Phys., **B431** (1994) 484, hep-th/9408099
- [2] A.Gorsky, I.Krichever, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., **B355** (1995) 466, hep-th/9505035
- [3] H.Itayama and A.Morozov, Nucl.Phys., **B491** (1997) 529, hep-th/9512161
- [4] A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Nucl.Phys., **B527** (1998) 690-716, hep-th/9802007
- [5] I.Krichever, Comm.Pure Appl.Math., **47** (1994) 437
- [6] B.Dubrovin, Nucl.Phys., **B379** (1992) 627; hep-th/9407018
- [7] E.Witten, Surv.Diff.Geom., **1** (1991) 243;
R.Dijkgraaf, E.Verlinde and H.Verlinde, Nucl.Phys., **B352** (1991) 59
- [8] A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., **B389** (1996) 43, hep-th/9607109; Mod.Phys.Lett., **A12** (1997) 773, hep-th/9701014
- [9] A.Marshakov, A.Mironov and A.Morozov, Int.J.Mod.Phys., **A12** (1997) 773, hep-th/9701123
- [10] K.Landsteiner, J.M.Pierre and S.B.Giddings, Phys.Rev., **D55** (1997) 2367, hep-th/9609059
- [11] A.Gorsky, S.Gukov and A.Mironov, Nucl.Phys., **B517** (1998) 409, hep-th/9707120
- [12] A.Klemm, W.Lerche, S.Theisen and S.Yankielowicz, Phys.Lett., **B344** (1995) 169, hep-th/9411048;
P.Argyres and A.Farragi, Phys.Rev.Lett., **74** (1995) 3931, hep-th/9411057;
A.Hanany and Y.Oz, Nucl.Phys., **B452** (1995) 283, hep-th/9505074
- [13] E.Martinec and N.Warner, Nucl.Phys., **B459** (1996) 97-112, hep-th/9509161
- [14] K.Landsteiner, E.Lopez and D.A.Lowe, JHEP 9807 (1998) 011, hep-th/9805158
- [15] I.Krichever and D.H.Phong, hep-th/9912180
- [16] R.Donagi and E.Witten, Nucl.Phys., **B460** (1996) 299, hep-th/9510101;
E.Martinec, Phys.Lett., **B367** (1996) 91, hep-th/9510204;
A.Gorsky and A.Marshakov, Phys.Lett., **B375** (1996) 127, hep-th/9510224
- [17] H.Itayama and A.Morozov, Nucl.Phys., **B477** (1996) 855, hep-th/9511125
- [18] H.W.Braden, A.Marshakov, A.Mironov and A.Morozov, Nucl.Phys., **B558** (1999) 371, hep-th/9902205
- [19] H.W.Braden, A.Marshakov, A.Mironov and A.Morozov, Nucl.Phys., **B573** (2000) 553, hep-th/9906240

- [20] A.Mironov and A.Morozov, Phys.Lett., **B475** (2000) 71, hep-th/9912092
- [21] A.Mironov and A.Morozov, hep-th/0001168
- [22] E.D'Hoker and D.H.Phong, Nucl.Phys., **B534** (1998) 697, hep-th/9804126
- [23] A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., **B380** (1996) 75, hep-th/9603140
- [24] A.Gorsky, A.Marshakov, A.Mironov and A.Morozov, hep-th/9604078
- [25] A.Gorsky, S.Gukov and A.Mironov, Nucl.Phys., **B518** (1998) 689, hep-th/9710239
- [26] A.Marshakov and A.Mironov, Nucl.Phys., **B518** (1998) 59-91, hep-th/9711156
- [27] A.Gorsky and A.Mironov, Nucl.Phys., **B550** (1999) 513, hep-th/9902030
- [28] E.Witten, Nucl.Phys., **B500** (1997) 3, hep-th/9703166
- [29] V.Inozemtsev, Comm.Math.Phys., **121** (1989) 629
- [30] V.E.Adler and A.B.Shabat, Theor.Math.Phys., **111** (1997) 323; **112** (1997) 179; I.Krichever, hep-th/9909224
- [31] A.Mironov, in: "Integrability: the Seiberg-Witten and Whitham equations", eds. H.W.Braden and I.M.Krichever, Amsterdam: Gordon and Breach science Publishers; hep-th/9903088
- [32] E.D'Hoker, I.Krichever and D.H.Phong, Nucl.Phys., **B489** (1997) 179
- [33] A.Beilinson and A.Levin, *Proceedings of Symposia in Pure Mathematics*, Vol.55, Part 2 (1994) 126-196
V.Kuznetsov, F.Nijhoff and E.Sklyanin, hep-th/9701004
J.Harvey and G.Moore, Nucl.Phys., **B463** (1996) 315-368, hep-th/9510182
- [34] E.D'Hoker and D.H.Phong, Nucl.Phys., **B513** (1998) 405, hep-th/9709053
- [35] H.W. Braden, A.Marshakov, A.Mironov and A.Morozov, Phys.Lett., **B448** (1999) 195, hep-th/9812078